TROPICAL GEOMETRY EXERCISES 5

(1) **(Dimension count.)**

(i) Show that a rational tropical curve of degree d in $\mathbb{R}^2$, with all vertices 3-valent and no marked points, has $3d-2$ vertices and $3d-3$ bounded edges.

(ii) Deduce that each combinatorial type of such curves has $3d-1$ parameters, and compare this with the classical count $d(d+3)/2-(d-1)(d-2)/2$.

(2) **(Conics have multiplicity one.)** The image of a 3-valent plane tropical curve of degree d is dual to a subdivision of $\Delta_d = \mathrm{Conv}\{(0,0),(d,0),(0,d)\}$ into triangles and parallelograms. There is one triangle for each vertex V, of normalised area $\mathrm{mult}(V)$, and one parallelogram for each self-crossing, of normalised area at least 2. Here areas are normalised so that the standard triangle Δ_1 has area 1, so the total normalised area is d^2.

(i) Deduce that a rational tropical conic counted in N_2^{trop} has all vertex multiplicities equal to 1 and no self-crossings.

(ii) (Optional) For cubics, show that either there is exactly one crossing and all multiplicities are 1, or there are no crossings and the multiplicities sum to 9.

Pencils. A plane curve of degree d is $V(F)$ for a nonzero homogeneous polynomial F of degree d, taken up to scaling. Its coefficients make these curves the points of a projective space $\mathbb{P}^N$, with $N = d(d+3)/2$.

- A *pencil* is a line in this $\mathbb{P}^N$. The pencil spanned by two curves $V(F) \neq V(G)$ of degree d is the family
$$V(\lambda F + \mu G), \qquad [\lambda : \mu] \in \mathbb{P}^1.$$
- Its *base points* are the points lying on every member, namely $V(F) \cap V(G)$. If F and G have no common factor, there are d^2 of them counted with multiplicity, by Bézout.
- Through a point p that is not a base point passes exactly one member of the pencil, since $\lambda F(p) + \mu G(p) = 0$ determines $[\lambda : \mu]$. So $p \mapsto$ (the member through p) is a map from $\mathbb{P}^2$ minus the base points to $\mathbb{P}^1$.
- Passing through a given point is one linear condition on $\mathbb{P}^N$. So the curves of degree d through k given points form a linear subspace of $\mathbb{P}^N$ of dimension at least $N-k$, and when this dimension is exactly 1 they form a pencil.

For example, the lines through a fixed point form a pencil, with that point as its only base point. The conics through four points $p_1, \dots, p_4$, no three on a line, form the pencil spanned by the line pairs $\overline{p_1p_2} \cup \overline{p_3p_4}$ and $\overline{p_1p_3} \cup \overline{p_2p_4}$, with base points $p_1, \dots, p_4$.

(3) **($N_2 = 1$.)** Recall that N_2 is the number of irreducible conics through 5 general points of $\mathbb{P}^2$.

(i) Show that the conics in $\mathbb{P}^2$ are parametrised by $\mathbb{P}^5$, and that the conics through a fixed point form a hyperplane in $\mathbb{P}^5$.

(ii) Let $p_1, \dots, p_5 \in \mathbb{P}^2$, no three on a line. Show that the five hyperplanes are linearly independent, so that exactly one conic passes through $p_1, \dots, p_5$.
Hint: otherwise the conics through the five points contain a pencil. Use it to find a conic through a sixth point of the line $\overline{p_1p_2}$, and apply Bézout to that line.

(iii) Show that this conic is irreducible, hence smooth, and that a smooth conic is isomorphic to $\mathbb{P}^1$ (project from a point on it). Conclude that $N_2 = 1$.

(4) **($N_3 = 12$, classically.)**

(i) Show that the cubics through 8 general points of $\mathbb{P}^2$ form a pencil, and that all of them pass through a 9th base point.

(ii) Blowing up the 9 base points gives $X \to \mathbb{P}^1$, whose fibres are the cubics of the pencil. Show that $\chi(X) = 12$.

(iii) Using $\chi(\text{smooth cubic}) = 0$ and $\chi(\text{nodal cubic}) = 1$, and assuming that the singular members of the pencil are irreducible and nodal, show that exactly 12 members are singular.