Lecture 6 :

Tropical Enumerative geometry -

The goal for the next 2/3 lectures is to introduce a tropical Curve Counting Problem, then show its solutions count certain geometric/algebraic Curves in $\mathbb{P}^2$.

Def: Let $C = V(F) \subset \mathbb{P}^2$ be a plane curve of degree d (So $F = F(x_0, x_1, x_2)$ homog. deg d).

Say C is irreducible

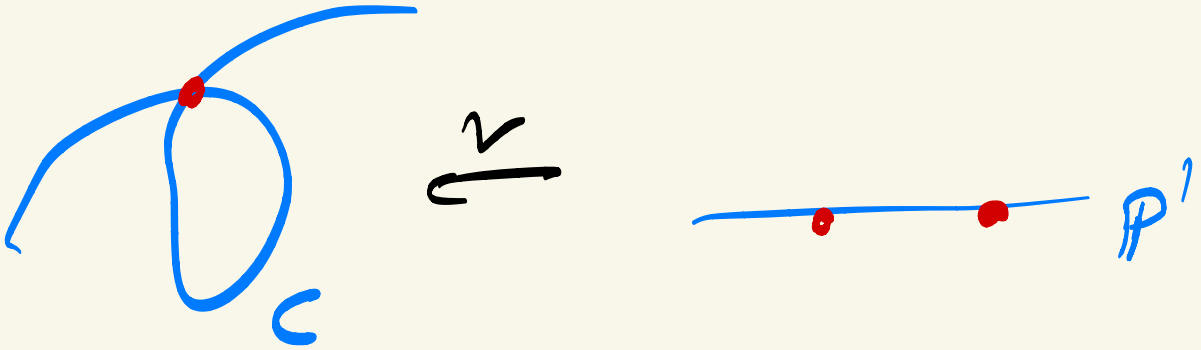
if $C \neq C_1 \cup C_2$, $C_i \subsetneq C$

subcurves.

Say C is rational if its

normalisation is $\mathbb{P}^1 \Leftrightarrow$ geometric genus of $C = 0$.

e.g.



Equivalently, C is the image of a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$

Ex.: Any line or conic is rational. A smooth cubic has genus 1 so not rational, but the nodal cubic $\mathcal{J}$ is.

$y^2 = x^2(x+1)$

Defn: let $N_d :=$ # irreducible rational curves through $3d-1$ general points in $\mathbb{P}^2$.

Fact: For general points, any such curve is nodal ie.

has Singular Points only of the form $(x^2 = y^2)$ locally.



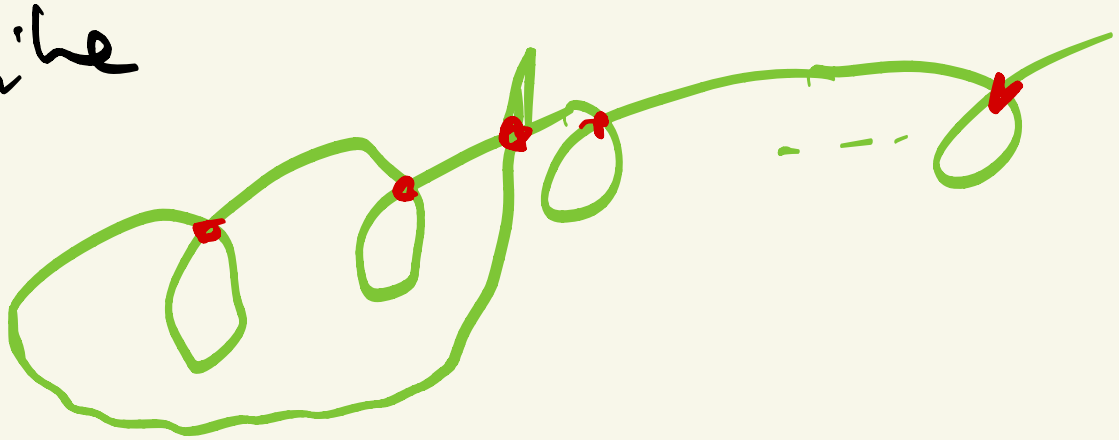
Q: why $3d-1$?

Lemma: Let C be nodal of degree d. Then C has precisely $\frac{(d-1)(d-2)}{2}$ nodes.

Proof: By the degree-genus formula, C has genus $\frac{1}{2}(d-1)(d-2)$.

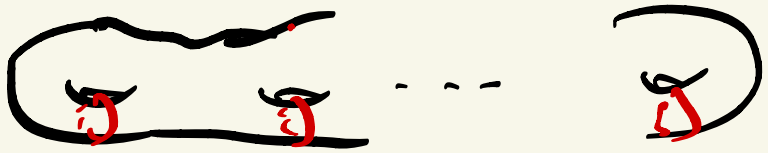
Since C is nodal, it looks tinned.

like



Each loop contributes $+1$ to genus and 1 node for every loop.

ie.



C'

$\Rightarrow$ Smooth genus $g = \frac{1}{2}(d-1)(d-2)$ ✓

C looks like C' after shrinking the g cycles to a point



1 node for each cycle. □

Back to the question of 3d-1:

Degree d curves form a Projective space $\mathbb{P}^{\frac{d}{2}(d+3)}$ via

$$C = V(F = \sum_{i+j+k=d} c_{ijk} x^i y^j z^k)$$

$$\longmapsto [c_{ijk}]_{ijk} \in \mathbb{P}^{\frac{d}{2}(d+3)}$$

e.g. $d=1$, $F = c_1x + c_2y + c_3z$

$\rightsquigarrow [c_1 : c_2 : c_3] \in \mathbb{P}^{2 = \frac{1}{2}(1+3)}$

let Z_δ = "δ-nodal curves" $\subset \mathbb{P}^{\frac{d}{2}(d+3)}$.

$$\operatorname{Codim}\left(Z_\delta, \mathbb{P}^{\frac{d}{2}(d+3)}\right) = \delta$$

(Roughly, e.g. $\delta = 1$ is given by vanishing of single equation coming from the Jacobian of F

$\leftrightarrow$ singular curves

$\rightsquigarrow$ 1-nodal curves $= V(\text{eqn})$

$\subseteq$ Codim 1.)

$$\Rightarrow \dim Z_\delta = \frac{d}{2}(d+3) - \delta$$

$$\delta = \frac{1}{2}(d-1)(d-2) \; ! \; \ldots = 3d-1 \; !$$

Now, C passes through $P = [x:y:z] \in \mathbb{P}^2$

$$\Leftrightarrow F(P) = 0$$

$$\Leftrightarrow [c_{ijk}]_{ijk} \in V\Big(\underbrace{\sum c_{ijk} x^i y^j z^k}_{\text{viewed as poly in the } c_{ijk}}\Big)$$

$$\cap$$

$$Z_\delta$$

ie, lies in a linear subspace.

Such a subspace is codim 1.

So in order to get a finite set we need

$$\#\text{eqns} = \#\text{fixed general pts}$$
$$= \dim \mathcal{I}_\delta = 3d-1.$$

Ex.: $N_1 = 1 = !$ line through 2 points

$N_2 = 1 = !$ conic through 5 points.

$N_3 = 12 = \#$ rational cubics through 8 pts (exercise).

Rmk:
Technically, we should write

$N_d(P_1, \dots, P_{3d-1})$ as there may be point dependence.

It will turn out to be indep.

for the $P_i \in \mathbb{P}^2$ general:

$$N_d = \#\left(Z_{3d-1} \cap H_1 \cap \dots \cap H_{3d-1}\right)$$

P_1, P_{3d-1}, I

$$= \deg Z_{3d-1} \text{ as a subvariety of } \mathbb{P}^{\frac{1}{2}d(d+3)}$$

which is indep. of the H_i.

§ The tropical enumerative Problem.

We now set up a tropical Curve counting Problem analogous to this algebraic one.

Defn: (a) An abstract tropical Curve with n legs is a metric tree Γ with vertices valence ≥ 3 and n legs labelled $x_1, \dots, x_n$.

(b) A Plane tropical Curve of degree d is a tuple

$(\Gamma, x_1, \ldots, x_n, h)$ where

$h : \Gamma \longrightarrow \mathbb{R}^2$ is continuous, affine on each edge with integer direction vectors $v(\vec{e})$ s.t.

(i) Balancing holds: $\sum_{\vec{e} \ni v} v(\vec{e}) = 0$.

(ii) marked ends are contracted ($v = 0$) and the other 3 d eds have directions $(-1, 0)$, $(0, -1)$, $(1, 1)$, 1 of each.

The set of such tropical curves also forms a $(3d-1)$-dimensional space: A 3-valent curve without markings has $3d-2$ vertices and $3d-3$ bounded edges (exercise!)



$d=1$

we have $+1$ parameter for each bounded edge, $+2$ parameters for choice of image of a vertex $\Rightarrow$ $3d-3+2=3d-1$.